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A Reduced-Form Model for Correlated Defaults with Regime-Switching Shot Noise Intensities

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Abstract In this paper, we consider a two-dimensional reduced form contagion model with regime-switching interacting default intensities. The model assumes that the intensities of the default times are driven by macro-economy described by a homogenous Markov chain and that the default of one firm may trigger a positive jump, associated with the state of Markov chain, in the default intensity of the other firm. The intensities before the default of the other firm are modeled by a two-dimensional regime-switching shot noise process with common shocks. By using the idea of "change of measure" and some closed-form formulas for the joint conditional Laplace transforms of the regime-switching shot noise processes and the integrated regime-switching shot noise processes, we derive the two-dimensional conditional and unconditional joint distributions of the default times. Based on these results, we can express the single-name credit default swap (CDS) spread, the first and second-to-default CDS spreads on two underlyings in terms of fundamental matrix solutions of linear, matrix-valued, ordinary differential equations.

Keywords Credit default swaps · Contagion model · Common shocks · Regime-switching · Shot noise intensities

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1 Introduction

Portfolio credit derivatives have attracted a lot of attention over the last decade in the credit risk theory. The challenge in the valuation of such financial derivatives is the modeling of the default dependence among them. Reduced form models are some of the most important ones dealing with correlated defaults. There are two major types of reduced form models for describing dependent default risk, namely bottom-up models and top-down models. In the former approach, one focuses on modeling default intensities of individual reference entities and their aggregation to form a portfolio default intensity. Some works on bottom-up models include Duffie and Gârleanu (2001), Jarrow and Yu (2001) and Giesecke and Goldberg (2004). In the latter approach, one concerns modeling default at portfolio level. A default intensity for the whole portfolio is modeled without reference to the constituent names. Some procedures such as random thinning can be used to recover the default intensities of the individual entities. Some works on top-down models include Brigo et al. (2007), Giesecke et al. (2011a) and Ding et al. (2009). We focus on bottom-up models.

To introduce default dependence under bottom-up models, one may set the default intensities of the firms in the portfolio to be driven by a common set of macroeconomic factors. Therefore, conditional on the realization of the macro-economic state variables, the default times are mutually conditionally independent. Default contagion is another approach to model the default correlation. The contagion models study the direct interaction of firms in which the default probability of one firm may change upon defaults of some other firms in the portfolio. They can well capture the clustering phenomena in correlated defaults and have been studied quite extensively in recent years, see for example, Kusuoka (1999), Davis and Lo (2001), Jarrow and Yu (2001), Ma and Yun (2010), and Yu (2007). Solving a contagion model faces an obstacle of looping default problem. Collin-Dufresne et al. (2004) propose a “change of measure” technique to deal with contagion models. This method is further extended by Giesecke and Zhu (2013), who develop an equivalent change of probability measure that includes the absolutely continuous measure change of Collin-Dufresne et al. (2004) as a limiting case. This paper aims to propose a model for correlated defaults which includes the above two mechanisms.

In finance, a point process with its intensity dependent on the point process itself or dependent on exogenous factors could provide a more effective model to capture the contagion phenomenon. See for example, Errais et al. (2010) analyze a family of multivariate affine point process models, in which the components of a multivariate affine point process are self- and cross-exciting, for applications in portfolio credit risk. The self-exciting specification can capture the feedback effects of events observed from real financial data. Another example of an intensity based model that incorporates feedback effects is in Ding et al. (2009), who propose a class of self-exciting loss processes that are obtained by time-changing a birth process. Errais et al. (2010) and Ding et al. (2009) mainly consider self-exciting processes. As pointed out by Errais et al. (2010), whether an affine process has the self-exciting property depends on the relation between the affine process and its intensity. The self-exciting property holds if the intensity depends on the affine process. Dassios and Jang (2003) consider an affine process having no self-exciting property, since the intensity they propose is only dependent on exogenous factors described by the shot-noise process. As explained by Dassios and Jang (2003), the shot noise process measures the frequency, magnitude and time period needed to go back to the previous level of intensity immediately after shock events occur. Recently, Gaspar and Schmidt (2010) show that an affine model augmented with shot-noise effects gives a superior fit to historical data as well as a better fit in calibration. However, the 2007–2010 global financial crisis reveals that default risk is much influenced by the business cycles or macro-economy. The above-mentioned models do not incorporate the change in regimes of credit markets. Intuitively, default risk typically declines during economic expansion because strong earnings keep overall defaults rates low. Default risk increases during economic recession because earnings deteriorate, making it more difficult to repay loans or make bond payments. Credit derivatives are long term instruments and thus it is very important to develop more appropriate models for valuation and risk management of credit products, which can take into account changes of market regimes or environments due to the crisis.

Markov regime-switching models have been used in different branches in modern financial economics, see for example, Elliott et al. (2005), Buffington and Elliott (2002), Hackbarth et al. (2006), Siu et al. (2008) and Siu (2010). In a regime switching model, the market is assumed to be in different states depending on the state of the macro-economy. Regime shift from one economic state to another may occur due to various financial factors like changes in business conditions, management decisions and other macro-economic conditions. Many papers have empirically verified the advantages of using the Markov regime-switching model. In the bond market, the switching behavior of market interest rate has been well documented in the empirical finance literature. For example, Ang and Bekaert (2002a, b) use interest rate data from the United States, Germany, and the United Kingdom to show empirically the switching behavior of market interest rates is attributed to business cycles. In the stock market, by using monthly returns data from the Standard and Poor's 500 and the Toronto Stock Exchange 300 indices, Hardy (2001) finds that the regime-switching lognormal model fits to the monthly returns data much better than other econometric models, such as the independent lognormal model and the ARCH type models. In the credit market, empirical studies point to the existence of different regimes in the default risk valuation, see for example, Davies (2004, 2008) and Giesecke et al. (2011b). Di Graziano and Rogers (2009) present a conditionally independent model, where defaults of different firms are driven by a common continuous-time Markov chain representing the state of health of the economy.

In this paper, motivated by Dassios and Jang (2003), Yu (2007), Di Graziano and Rogers (2009) and others, we propose a two-dimensional contagion model under a Markov, regime-switching environment, where the intensities are driven by a common continuous-time Markov chain representing the state of the macro-economy as well as the default of the other firm, and the intensities before the default of the other firm are assumed to follow regime-switching shot noise processes. Therefore, the default dependence structure we construct includes both mechanisms: conditional on common macro-economy and default contagion. As explained in Dassios and Jang (2003), the intensity changes upon the arrival of the shock events when the intensity is modeled by a regime-switching shot noise process. Intuitively, the default intensities may be both affected by some common shock events. Following Lindskog and McNeil (2003), we propose a two-dimensional regime-switching shot noise process with common shocks to model the intensities before the default of the other firm.

The aim of this paper is to provide a model for correlated defaults. Under the proposed model, we will derive the joint distribution of the default times, the single-name CDS spreads with and without counterparty risk, the first and second-to-default CDS spreads on two underlyings. The paper is organized as follows: Section 2 introduces the default dependence structure under a Markov, regime-switching environment and presents some preliminary results. In Section 3, we use the idea of "change of measure" proposed by

Collin-Dufresne et al. (2004) and further studied by Giesecke and Zhu (2013) to derive the two-dimensional conditional and unconditional joint distributions of the default times. Based on the results, we give the formulas for the single-name CDS spreads with and without counterparty risk, the first and second-to-default CDS spreads on two underlyings in Section 4. Section 5 performs some numerical calculations. Section 6 concludes.

2 Modeling Default Dependence Under a Markov Environment

In this section, we propose a two-dimensional contagion model under a Markov environment within the reduced form framework. Consider a continuous-time model with a finite time horizon $[0, T]$ with $T < \infty$. Let $\{\Omega, \Im, \{\Im_t\}_{0\le t\le T}, P\}$ be a filtered complete probability space, where P is the risk neutral measure and $\{\Im_t\}_{0\le t\le T}$ is a filtration satisfying the usual conditions of right continuity and completeness. Throughout the paper, it is assumed that all random variables and stochastic processes are well defined on this probability space and $\Im_T-$measurable.

Let $\{X_t\}_{t\ge 0}$ be a homogenous continuous-time irreducible Markov chain with generator $Q = (q_{ij})_{i,j=1,2,\cdots,N}$, generating a filtration $\Im_t^X$. As in Buffington and Elliott (2002), the state space of X can be taken to be, without loss of generality, the set of unit vectors $\{e_1, e_2, \cdots, e_N\}$, $e_i = (0, \cdots, 0, 1, 0, \cdots, 0)^* \in R^N$, where $*$ denotes the transpose of a vector or a matrix. Elliott et al. (1994) provide the following semi-martingale decomposition for $\{X_t\}_{t\ge 0}$:

$$X_t = X_0 + \int_0^t Q^* X_s ds + M_t, \tag{2.1}$$

where $\{M_t\}_{t\ge 0}$ is an R^N-valued martingale with respect to the filtration generated by $\{X_t\}_{t\ge 0}$.

Let $\langle ., .\rangle$ denote a scalar product in R^N, that is, for any $\mathbf{x}, \mathbf{y} \in R^N$, $\langle \mathbf{x}, \mathbf{y}\rangle = \sum_{i=1}^N x_i y_i$. Denote by $D(t_1, t_2)$ the stochastic factor giving the discounted value of one at time t_1 due at time t_2. Assume that the discount factor in this paper is given by $D(0, t) = \exp\{-\int_0^t r_s ds\}$, where the stochastic interest rate is modeled by $r_t = \langle \mathbf{r}, X_t\rangle$, for a vector $\mathbf{r} = (r_1, r_2, \cdots, r_N)^* \in R^N$ with $r_i > 0$ for each $i = 1, 2, \cdots, N$.

Now we model the default dependence structure under a Markov, regime-switching environment. Denote by τ_1, τ_2 the default times of two firms. Define τ_i as

$$\tau_i = \inf\left\{t > 0 : \int_0^t \lambda_s^i ds \ge E_i\right\}, i = 1, 2, \tag{2.2}$$

where E_1 and E_2 are two independent unit exponential random variables, and for each $i = 1, 2$, λ_t^i is the nonnegative $\Im_t$ predictable intensity of τ_i satisfying $E[\int_0^t \lambda_s^i ds] < \infty$, for any $t < +\infty$. Assume that the default intensities of two firms are expressed as

$$\begin{cases} \lambda_t^1 = L_t^1 + a_{\tau_2}^1 e^{-\delta^1(t-\tau_2)} 1_{\{\tau_2\le t\}}, \\ \lambda_t^2 = L_t^2 + a_{\tau_1}^2 e^{-\delta^2(t-\tau_1)} 1_{\{\tau_1\le t\}}, \end{cases} \tag{2.3}$$

where $a_{\tau_i}^j = \langle \mathbf{a}^j, X_{\tau_i}\rangle$ for a constant vector $\mathbf{a}^j = (a^{j1}, \cdots, a^{jN})^*$ with $a^{ji} > 0$ for each $j = 1, 2, i = 1, \cdots, N$, and L_t^i is a regime-switching shot noise process given by

$$L_t^i = L_0^i e^{-\delta^i t} + \int_0^t e^{-\delta^i(t-s)} dJ_s^i, i = 1, 2. \tag{2.4}$$

Here δ^1, δ^2 are positive constants; $L_0^i = \langle \mathbf{L}_0^i, X_0 \rangle$, where $\mathbf{L}_0^i = (L_0^{i1}, \cdots, L_0^{iN})^*$ with $L_0^{ij} > 0$ for each $i = 1, 2$, $j = 1, 2, \cdots, N$; and $J_t^i = \sum_{j=1}^{N_i(t)+N_3(t)} Y_j^i$, where $N_1(t)$, $N_2(t)$ and $N_3(t)$ are mutually conditionally independent regime-switching Poisson processes with intensities given by $\rho_i(s) = \langle \boldsymbol{\rho}_i, X_s \rangle$ for constant vectors $\boldsymbol{\rho}_i = (\rho_i^1, \cdots, \rho_i^N)^*$, $i = 1, 2, 3$ with $\rho_i^j > 0$, for each $i = 1, 2, 3$, $j = 1, \cdots, N$; Assume that given the path of the Markov chain X, the two sequences $\{Y_1^1, Y_2^1, \cdots\}$, $\{Y_1^2, Y_2^2, \cdots\}$ are independent and independent of $N_1(t), N_2(t), N_3(t)$. Furthermore, given the path of the Markov chain X, we assume that for each $i = 1, 2$, the jump sizes Y_j^i, $j = 1, 2, \cdots$ are mutually independent and identically distributed with a common conditional density f_t^i concentrated on $(0, \infty)$, where $f_t^i(.) = \langle \mathbf{f}^i(.), X_t \rangle$, with $\mathbf{f}^i(.) = (f^{i1}(.), \cdots, f^{iN}(.))^*$. If there is no regime-switching, then the dependence structure between J_t^1 and J_t^2 is so-called "common shock structure" discussed by Cossette and Marceau (2000). Therefore, the process (L_t^1, L_t^2) is a two-dimensional regime-switching shot noise process with common jumps.

Note that, from Eqs. 2.3 and 2.4, the default intensities of the two firms satisfy the SDEs

$$\begin{cases} d\lambda_t^1 = -\delta^1 \lambda_t^1 dt + dJ_t^1 + a_t^1 d\overline{N}_t^2, \\ d\lambda_t^2 = -\delta^2 \lambda_t^2 dt + dJ_t^2 + a_t^2 d\overline{N}_t^1, \end{cases} \tag{2.5}$$

where $\overline{N}_t^i = 1_{\{\tau_i \le t\}}$, $i = 1, 2$. Comparing with the affine process studied by Errais et al. (2010), we incorporate the changes of market regimes or environments due to the crisis into the the dynamics of stochastic intensities. Therefore, the process $(\overline{N}_t^1, \overline{N}_t^2)^*$, driven by a two-dimensional regime-switching affine jump process $(\lambda_t^1, \lambda_t^2)^*$, is a two-dimensional regime-switching affine process. Although Errais et al. (2010) present a formula for the transform of a general affine point process, Although Errais et al. (2010) present a formula for the transform of a general affine point process, it can not be applied to the Markov, regime-switching intensity model we propose. In this paper, we shall derive a formula for the Laplace transform of L_t^i. Then by using the idea of "change of measure," we can derive the default probabilities and the CDS spread.

This paper introduces a new point process with regime switching by generalising an affine process studied in Errais et al. (2010). Our process includes both self-excited and externally excited jumps. Note that, the default dependence modeled by Eq. 2.3 stems from three sources. First, the intensities of the two firms are both affected by macro-economic factors, so we have inter-dependence between their defaults through a Markov chain, which describes the macro-economy. This dependence structure allows the default intensities of the two firms to change simultaneously over time depending on the state of the underlying Markov chain. Second, default dependence arises from common jumps in the intensities modeled by regime-switching compound Poisson process. Finally, inter-dependent default structure arises from default contagion. Broadly speaking, there are two kinds of default contagion, namely, counterparty risk and information effect. If there exist direct business links between the two firms, such as an intense business relation or a strong borrower-lender relationship, then these direct links lead to default contagion and counterparty risk. So the conditional default probability of non-defaulted firm given the additional information that some other firm has defaulted is higher than the unconditional default probability, and the credit spread of the bond issued by non-defaulted firm increases given the news that some other firm has defaulted. In mathematical terms default contagion and counterparty risk lead to an upward jumps in the default intensity of the non-defaulted firm at the default time of some other firm. However, there is another kind of default contagion, information effect. That is, changes in the conditional default probability of the non-defaulted firm can be caused by information effects: investors might revise their estimate of the financial health of the non-defaulted firm in light of the news that a particular firm has defaulted. See for example, the accounting scandal of the WorldCom led to rising credit spreads for many other corporations. Furthermore, our model can explain the following important issues: credit spreads may change without default occurring and credit spreads exhibit both a jump and a continuous component.

Remark 2.1 If the jump amounts $\mathbf{a}^j$, $j = 1, 2$ are set to be 0 and J_t^1, J_t^2 are assumed to be independent given X, then the default times are conditionally independent. Therefore, Eq. 2.3 becomes to be a conditionally independent default model.

3 Conditional Joint Distribution and Laplace Transform

In this section, we follow the idea of change of measure to derive the two-dimensional conditional distributions of the default times, and we use a martingale method to obtain the joint Laplace transform of the regime-switching shot noise processes.

Denote the filtration by

$$\Im_t = \Im_t^X \vee \Im_t^L \vee \Im_t^1 \vee \Im_t^2,$$

where $\Im_t^L = \Im_t^{L^1} \vee \Im_t^{L^2}$, and $\Im_t^i = \sigma\left(H_u^i : 0 \leq u \leq t\right)$, with $H_u^i = 1_{\{\tau_i \leq u\}}$, $\Im_t^{L^i} = \sigma\left(L_u^i : 0 \leq u \leq t\right), i = 1, 2$.

For the computation of the joint distribution of τ_1 and τ_2, the main obstacle is the looping structure Eq. 2.3 of the intensities. Following Collin-Dufresne et al. (2004), we define the following two survival measures

$$\frac{dP^i}{dP}|_{\Im_t} = 1_{\{\tau_i > t\}} \exp\left(\int_0^t \lambda_s^i ds\right) \doteq \eta_t^i, i = 1, 2, \tag{3.1}$$

where P^i is a firm-specific (firm i) probability measure which is absolutely continuous with respect to P on the stochastic interval $[0, \tau_i)$. From Lemma A.2 in Collin-Dufresne et al. (2004), we have $1_{\{\tau_i > t\}} \exp\left(\int_0^t \lambda_s^i ds\right)$ is a uniformly integrable P-martingale with respect to $\Im_t$ and is almost surely strictly positive on $[0, \tau_i)$ and almost surely equal to zero on $[\tau_i, \infty)$. To proceed the calculations under the measure P^i, we enlarge the filtration to $\overline{\Im}^i = \left(\overline{\Im}_t^i\right)_{t \geq 0}$ as the completion of $\Im = (\Im_t)_{t \geq 0}$ by the null sets of the probability measure P^i. Let $E^i[.]$ denote the expectation taken under the measure P^i. For notational convenience, we still use $\Im$ instead of $\overline{\Im}^1$ or $\overline{\Im}^2$ used without changing the results. The next results show that, under P^i, the Markov chain X_t and the jump process L_t^i have the same distributions as those under P.

Proposition 3.1 *The process*

$$M_t = X_t - X_0 - \int_0^t Q^* X_s ds,$$

is an R^N-valued martingale under P^i.

Proof The proof is presented in the Appendix. □

Proposition 3.2 *For* $i = 1, 2$, *the processes*

$$\overline{N}_i(t) = N_i(t) - \int_0^t \rho_i(s)ds,$$

and

$$\overline{M}_i(t) = J_t^i - \int_0^t \int_0^\infty (\rho_i(s) + \rho_3(s)) y f_s(y) dy ds,$$

are both R^N*-valued martingales under* P^i.

Proof The proof is presented in the Appendix. □

Therefore, from Propositions 3.1 and 3.2, we can conclude that the distributions of L_t^i, X_t under the measure P^i are the same as those under P.

The next two results give the conditional joint distributions of the default times by using the change of measure technique.

Proposition 3.3 *For* $0 < s \le T$,

$$P\left(\tau_1 > s, \tau_2 > s \,\middle|\, \Im_s^X \vee \Im_s^L\right) = e^{-\int_0^s (L_u^1 + L_u^2) du}; \tag{3.2}$$

for $0 \le t < s \le T$,

$$P\left(\tau_1 > s, t < \tau_2 \le s \,\middle|\, \Im_s^X \vee \Im_s^L\right) = \int_t^s L_v^2 e^{-\int_0^v L_u^2 du - \int_0^s L_u^1 du - a_v^1 \xi^1(v,s)} dv; \tag{3.3}$$

and

$$P\left(\tau_2 > s, t < \tau_1 \le s \,\middle|\, \Im_s^X \vee \Im_s^L\right) = \int_t^s L_v^1 e^{-\int_0^v L_u^1 du - \int_0^s L_u^2 du - a_v^2 \xi^2(v,s)} dv, \tag{3.4}$$

where $\xi^i(v, s) = (1 - \exp(-\delta^i(s - v)))/\delta^i$, $i = 1, 2$.

Proof The proof is presented in the Appendix. □

Now we turn to deriving the Laplace transform of the regime-switching shot noise processes which plays an important role in the valuation of the CDS spreads.

For $c^i \ge 0$ and $d^i > 0$, $i = 1, 2$, let

$$V(t, T) = E\left[e^{-\int_t^T \left(\sum_{i=1}^2 c^i L_s^i + r_s\right) ds - \sum_{i=1}^2 d^i L_T^i} X_T \,\middle|\, \Im_t^X \vee \Im_t^L \right],$$

where L_t^i is modeled by Eq. 2.4. Note that $L_t^i > 0$ for $i = 1, 2$. Consequently, $V(t, T)$ is a bounded vector. Since $\left(X_t, L_t^1, L_t^2\right)^*$ is a three-dimensional Markov process with respect to $\Im_t^X \vee \Im_t^L$, we have

$$V(t, T) = E\left[e^{-\int_t^T \left(\sum_{i=1}^2 c^i L_s^i + r_s\right) ds - \sum_{i=1}^2 d^i L_T^i} X_T \,\middle|\, L_t^1, L_t^2, X_t \right] =: \theta\left(t, T, L_t^1, L_t^2, X_t\right).$$

In particular, given that $L_t^i = l^i$ for $i = 1, 2$ and $X_t = x$,

$$\theta(t, T, l^1, l^2, x) = E\left[e^{-\int_t^T \left(\sum_{i=1}^{2} c^i L_s^i + r_s \right) ds - \sum_{i=1}^{2} d^i L_T^i} X_T \,\middle|\, L_t^1 = l^1, L_t^2 = l^2, X_t = x \right].$$

Write

$$\theta_i = \theta(t, T, l^1, l^2, e_i), i = 1, 2, \cdots, N, \ \boldsymbol{\theta} = (\theta_1, \theta_2, \cdots, \theta_N)^* \in R^N.$$

Assume that $\theta(t, T, l^1, l^2, x)$ is continuously differentiable with respect to t and l^i. Denote the corresponding derivatives by $\partial\theta/\partial t$ and $\partial\theta/\partial l^i$ for $i = 1, 2$. The following result gives the explicit expression for $\theta(t, T, L_t^1, L_t^2, X_t)$.

Proposition 3.4 *For $c^i \geq 0$ and $d^i > 0$, $i = 1, 2$, we have*

$$V(t, T) = e^{-\sum_{i=1}^{2} (c^i \xi^i (t,T) + d^i e^{-\delta^i (T-t)}) L_t^i} \langle \mathbf{A}_1(c^1, c^2, d^1, d^2, t, T), X_t \rangle, \tag{3.5}$$

where

$$\xi^i(t, T) = (1 - e^{-\delta^i (T-t)})/\delta^i, i = 1, 2,$$

and the matrix $\mathbf{A}_1(c^1, c^2, d^1, d^2, t, T)$ solves

$$\frac{\partial \mathbf{A}_1}{\partial t} + (Q - \mathbf{diag}(\mathbf{r} - \mathbf{F}_t(c^1, c^2, d^1, d^2))\mathbf{A}_1(c^1, c^2, d^1, d^2, t, T) = 0,$$

with boundary condition

$$\mathbf{A}_1(c^1, c^2, d^1, d^2, T, T) = \mathbf{I}.$$

Here, $\mathbf{I}$ is an $N \times N$ identity matrix and $\mathbf{F}_s$ is an N-dimensional vector with the jth component given by

$$F_s^j(c^1, c^2, d^1, d^2) = \sum_{i=1}^{2} \rho_i^j \left(g_s^{ij}(c^i, d^i) - 1 \right) + \rho_3^j \left(\prod_{i=1}^{2} g_s^{ij}(c^i, d^i) - 1 \right),$$

and

$$g_s^{ij}(c^i, d^i) = \int_0^\infty e^{-(c^i \xi^i (s,T) + d^i e^{-\delta^i (T-s)})x} f^{ij}(x) dx, \qquad i = 1, 2, \quad j = 1, \cdots, N.$$

And

$$E\left[e^{-\int_t^T \left(\sum_{i=1}^{2} c^i L_s^i + r_s \right) ds - \sum_{i=1}^{2} d^i L_T^i} \,\middle|\, \Im_t^X \vee \Im_t^L \right]$$
$$= e^{-\sum_{i=1}^{2} (c^i \xi^i (t,T) + d^i e^{-\delta^i (T-t)}) L_t^i} \langle \mathbf{A}_1(c^1, c^2, d^1, d^2, t, T)\mathbf{1}, X_t \rangle. \tag{3.6}$$

Proof We use the martingale approach to derive Eq. 3.5. Consider the function $\overline{V}(t,T) = U_t\theta(t,T,L_t^1,L_t^2,X_t)$, where $U_t = \exp\left(-\int_0^t\left(\sum_{i=1}^{2} c^i L_s^i + r_s\right)ds\right)$. Applying Itô's differentiation rule to $\overline{V}(t,T)$ yields

$$
\begin{aligned}
d\overline{V}(t,T) \;=\; & U_t\left(\frac{\partial}{\partial t} - \sum_{i=1}^{m}\delta^i L_t^i\frac{\partial}{\partial l^i} - \left(\sum_{i=1}^{2} c^i L_t^i + r_t\right)\right)\theta(t,T,L_t^1,L_t^2,X_t)dt\\
&+ U_t((\theta(t,T,L_t^1,L_t^2,X_t) - \theta(t,T,L_{t-}^1,L_t^2,X_t))dN_t^1\\
&+ U_t((\theta(t,T,L_t^1,L_t^2,X_t) - \theta(t,T,L_t^1,L_{t-}^2,X_t))dN_t^2\\
&+ U_t(\theta(t,T,L_t^1,L_t^2,X_t) - \theta(t,T,L_{t-}^1,L_{t-}^2,X_t))dN_t^3\\
&+ U_t\langle\boldsymbol{\theta}, Q^*X_t\rangle dt + U_t\langle\boldsymbol{\theta}, dM_t\rangle.
\end{aligned}
$$

Note that $\overline{V}(t,T)$ is a bounded martingale. Consequently, we have

$$
\begin{aligned}
&\left(\frac{\partial}{\partial t} - \sum_{i=1}^{2}\delta^i l^i\frac{\partial}{\partial\lambda^i} - \left(\sum_{i=1}^{m} c^i\lambda^i + \langle\mathbf{r},x\rangle\right)\right)\theta(t,T,l^1,l^2,x) + \langle\boldsymbol{\theta}, Q^*x\rangle\\
&+\ \langle\boldsymbol{\rho_1},x\rangle E[(\theta(t,T,l^1+Y^1,l^2,x) - \theta(t,T,l^1,l^2,x))]\\
&+\ \langle\boldsymbol{\rho_2},x\rangle E[(\theta(t,T,l^1,l^2+Y^2,x) - \theta(t,T,l^1,l^2,x))]\\
&+\ \langle\boldsymbol{\rho_3},x\rangle E[(\theta(t,T,l^1+Y^1,l^2+Y^2,x) - \theta(t,T,l^1,l^2,x))] = 0.
\end{aligned}
\tag{3.7}
$$

Due to the affine structure of L_t^i for $i = 1,2,$ motivated by Duffie et al. (2003), we try the solution

$$
\theta(t,T,l^1,l^2,x) = e^{\sum_{i=1}^{2} B_i(t,T)l^i}\,C(t,T,x), \tag{3.8}
$$

where the terminal conditions are given by

$$
B_i(T,T) = -d^i,\ C(T,T,x) = x.
$$

Write $\mathbf{C}(t,T) = (C(t,T,e_1),\cdots,C(t,T,e_N))^* \in \mathbf{R^N}$. Substituting the solution for θ given by Eq. 3.8 into 3.7 gives

$$
\begin{aligned}
&C(t,T,x)\sum_{i=1}^{2} l^i\left(\frac{\partial B_i}{\partial t} - \delta^i B_i(t,T) - c^i\right) + \frac{\partial C}{\partial t} - \langle\mathbf{r},x\rangle C(t,T,x)\\
&+\ \langle\mathbf{C}(t,T), Q^*x\rangle + C(t,T,x)\sum_{i=1}^{2}\langle\boldsymbol{\rho_i},x\rangle\int_0^\infty (e^{B_i(t,T)y} - 1)\langle\mathbf{f}^i(y),x\rangle dy\\
&+\ C(t,T,x)\langle\boldsymbol{\rho_3},x\rangle\left(\prod_{i=1}^{2}\int_0^\infty e^{B_i(t,T)y}\langle\mathbf{f}^3(y),x\rangle dy - 1\right) = 0.
\end{aligned}
\tag{3.9}
$$

Since Eq. 3.9 holds for all l^i and x, we have

$$
\frac{\partial B_i}{\partial t} - \delta^i B_i(t,T) - c^i = 0,\ B_i(T,T) = -d^i, \qquad i = 1,2,
$$

and

$$
\frac{\partial\mathbf{C}}{\partial t} + (Q + \mathbf{diag}(\overline{\mathbf{F}}_t - \mathbf{r}))\mathbf{C}(t,T) = 0,\ \mathbf{C}(T,T) = \mathbf{I},
$$

where $\overline{\mathbf{F}}_t$ is an N-dimensional vector with the jth component given by

$$\overline{F}_t^j = \sum_{i=1}^{2} \rho_i^j \int_0^\infty (e^{B_i(t,T)y} - 1) f^{ij}(y)dy + \rho_0^3 \left(\prod_{i=1}^{2} \int_0^\infty e^{B_i(t,T)y} f^{ij}(y)dy - 1 \right).$$

By solving the above equations, we complete the proof of Eq. 3.5.

Equation 3.6 holds since $E\left[e^{-\int_t^T \left(\sum_{i=1}^{2} c^i L_s^i + r_s \right) ds - \sum_{i=1}^{2} d^i L_T^i} | \Im_t^X \vee \Im_t^L \right] = \langle V(t,T), \mathbf{1} \rangle.$ □

Corollary 3.1 *For $c^i \geq 0$ and $d^i > 0$, $i = 1, 2$ and for each $k = 1, 2$, we have*

$$E\left[L_T^k e^{-\int_t^T \left(\sum_{i=1}^{2} c^i L_s^i + r_s \right) ds - \sum_{i=1}^{2} d^i L_T^i} X_T \middle| \Im_t^X \vee \Im_t^L \right] = e^{-\sum_{i=1}^{2} (c^i \xi^i(t,T) + d^i e^{-\delta^i (T-t)}) L_t^i}$$

$$\times \langle e^{-\delta^k (T-t)} L_t^k \mathbf{A}_1(c^1, c^2, d^1, d^2, t, T) - \mathbf{A}_2^k(c^1, c^2, d^1, d^2, t, T), X_t \rangle, \tag{3.10}$$

and

$$E\left[L_T^1 L_T^2 e^{-\int_t^T \left(\sum_{i=1}^{2} c^i L_s^i + r_s \right) ds - \sum_{i=1}^{2} d^i L_T^i} X_T \middle| \Im_t^X \vee \Im_t^L \right] = e^{-\sum_{i=1}^{2} (c^i \xi^i(t,T) + d^i e^{-\delta^i (T-t)}) L_t^i}$$

$$\times \langle e^{-(\delta^1+\delta^2)(T-t)} L_t^1 L_t^2 \mathbf{A}_1(c^1, c^2, d^1, d^2, t, T) - e^{-\delta^1 (T-t)} L_t^1 \mathbf{A}_2^2(c^1, c^2, d^1, d^2, t, T)$$
$$- e^{-\delta^2 (T-t)} L_t^2 \mathbf{A}_2^1(c^1, c^2, d^1, d^2, t, T) + \mathbf{A}_3^{12}(c^1, c^2, d^1, d^2, t, T), X_t \rangle, \tag{3.11}$$

where

$$\mathbf{A}_2^l(c^1, c^2, d^1, d^2, t, T) = \frac{\partial \mathbf{A}_1(c^1, c^2, d^1, d^2, t, T)}{\partial d^l}, l = 1, 2,$$

and

$$\mathbf{A}_3^{12}(c^1, c^2, d^1, d^2, t, T) = \frac{\partial^2 \mathbf{A}_1(c^1, c^2, d^1, d^2, t, T)}{\partial d^1 \partial d^2},$$

with $\mathbf{A}_1(c^1, c^2, d^1, d^2, t, T)$ defined in Proposition 3.4.

Proof If we differentiate both sides of Eq. 3.5 with respect to d^k, then we can obtain Eq. 3.10. Also, Eq. 3.11 can be obtained by taking partial derivatives with respect to d^1 and then d^2 on both sides of Eq. 3.5. □

For notational convenience, define

$$\mathbf{A}_2^i(c^1, c^2, 0, d^2, t, T) = \lim_{d^1 \to 0} \mathbf{A}_2^i(c^1, c^2, d^1, d^2, t, T),$$
$$\mathbf{A}_2^i(c^1, c^2, d^1, 0, t, T) = \lim_{d^2 \to 0} \mathbf{A}_2^i(c^1, c^2, d^1, d^2, t, T),$$
$$\mathbf{A}_2^i(c^1, c^2, 0, 0, t, T) = \lim_{d^1, d^2 \to 0} \mathbf{A}_2^i(c^1, c^2, d^1, d^2, t, T),$$
$$\mathbf{A}_3^{12}(c^1, c^2, 0, d^2, t, T) = \lim_{d^1 \to 0} \mathbf{A}_3^{12}(c^1, c^2, d^1, d^2, t, T),$$
$$\mathbf{A}_3^{12}(c^1, c^2, d^1, 0, t, T) = \lim_{d^2 \to 0} \mathbf{A}_3^{12}(c^1, c^2, d^1, d^2, t, T),$$

where $\mathbf{A}_2^i(c^1, c^2, d^1, d^2, t, T)$ and $\mathbf{A}_3^{12}(c^1, c^2, d^1, d^2, t, T)$ are defined in Corollary 3.1.

4 Credit Default Swaps

In recent years, the credit derivatives market has grown explosively and credit derivatives have become popular tools for hedging credit risk of financial institutions. A credit derivative may be reference to a single reference entity, or a portfolio of reference entities. In this section, we shall compute the fair spreads of a single-name credit default swap, the first and second-to-default basket swaps on two underlyings.

4.1 Single-Name Credit Default Swap

In this subsection, we shall compute the fair credit default swap premiums with and without default risk of the protection seller, respectively.

A single name CDS is an insurance contract on the default of a single reference credit between a protection buyer and a protection seller. We assume the protection buyer is default-free. Consider a CDS contract with notional value one, continuous spread rate payments and maturity T. Indices 1, 2 refer to quantities related to the protection seller and the reference entity. Denote by τ_1 and τ_2 the default times of the protection seller and the reference entity, respectively, denote by R the recovery of the reference entity which is supposed to be a constant. Assume that the default intensities of τ_1 and τ_2 are given by Eq. 2.3. Let κ and κ_1 be the fair spreads of a CDS contract without and with the default risk of the protection seller, respectively. In the literature, much research has been carried out to study the impact of counterparty risk on CDS valuation. In this paper, the impact on the CDS spread rate in the presence of the counterparty risk is measured by $\kappa - \kappa_1$, which has also been studied in Leung and Kwok (2009).

We first describe the cash flows of a CDS without counterparty. For the default leg, the protection seller covers the credit losses $1 - R$ as soon as the reference entity has defaulted. For the premium leg, the protection buyer pays κ to the seller continuously until maturity or until the reference entity defaults before maturity. Then, the fair spread of the CDS without counterparty risk is determined so that the discounted payoff of the two legs are equal when the contract is initiated at time 0. That is, the spread κ should satisfy

$$\kappa \int_0^T E\left[1_{\{\tau_2>u\}} D(0,u)\right] du = (1-R) E\left[D(0,\tau_2) 1_{\{\tau_2 \le T\}}\right].$$

Hence,

$$\kappa = \frac{(1-R) E\left[D(0,\tau_2) 1_{\{\tau_2 \le T\}}\right]}{\int_0^T E\left[1_{\{\tau_2>u\}} D(0,u)\right] du}. \tag{4.1}$$

We now turn to the cash flows of a CDS with counterparty risk. For the default leg, if the reference entity defaults before maturity while the protection seller still survives, then the protection seller covers the credit losses $1 - R$. For simplicity, we assume that if the protection seller defaults first before maturity, then the protection buyer gets nothing. For the premium leg, the protection buyer pays κ_1 to the seller continuously until maturity or until any of names 1, 2 defaults before maturity. Again, the fair spread of the CDS with counterparty risk is determined so that the discounted payoff of the two legs are equal when the contract is initiated at time 0. So, the spread κ_1 should satisfy

$$\kappa_1 \int_0^T E\left[1_{\{\tau_1 \wedge \tau_2 > u\}} D(0,u)\right] du = (1-R) E\left[D(0,\tau_2) 1_{\{\tau_2 \le T, \tau_2 < \tau_1\}}\right].$$

So,

$$\kappa_1 = \frac{(1-R)E\left[D(0,\tau_2)1_{\{\tau_2 \le T, \tau_2 < \tau_1\}}\right]}{\int_0^T E\left[1_{\{\tau_1 \wedge \tau_2 > u\}} D(0,u)\right] du}. \tag{4.2}$$

Proposition 4.1 *The fair spread of the CDS without counterparty risk is given by*

$$\kappa = \frac{(1-R)\int_0^T (h_3(t)+h_4(t))dt}{\int_0^T (h_1(t)+h_2(t))dt}, \tag{4.3}$$

where

$$h_1(t) = e^{-\xi^1(0,t)L_0^1 - \xi^2(0,t)L_0^2} \langle \mathbf{A}_1(1,1,0,0,0,t)\mathbf{1}, X_0\rangle,$$

$$h_2(t) = \int_0^t e^{-\xi^1(0,v)L_0^1 - \xi^2(0,t)L_0^2} \langle (e^{-\delta^1 v} L_0^1 \mathbf{A}_1(1,1,0,\xi^2(v,t),0,v)$$
$$-\mathbf{A}_2^1(1,1,0,\xi^2(v,t),0,v))\mathbf{U}_2(v,t)(\mathbf{A}_1(0,1,0,0,v,t)\mathbf{1}), X_0\rangle dv,$$

$$\begin{aligned}
h_3(t) = \int_0^t & e^{-\xi^1(0,v)L_0^1 - \xi^2(0,t)L_0^2} (\langle (e^{-(\delta^1+\delta^2)v} L_0^1 L_0^2 \mathbf{A}_1(1,1,0,\xi^2(v,t),0,v) \\
& - e^{-\delta^1 v} L_0^1 \mathbf{A}_2^2(1,1,0,\xi^2(v,t),0,v) - e^{-\delta^2 v} L_0^2 \mathbf{A}_2^1(1,1,0,\xi^2(v,t),0,v) \\
& + \mathbf{A}_3^{12}(1,1,0,\xi^2(v,t),0,v)) e^{-\delta^2(t-v)} \mathbf{U}_2(v,t)(\mathbf{A}_1(0,1,0,0,v,t)\mathbf{1}), X_0\rangle \\
& + \langle (e^{-\delta^1 v} L_0^1 \mathbf{A}_1(1,1,0,\xi^2(v,t),0,v) - \mathbf{A}_2^1(1,1,0,\xi^2(v,t),0,v)) \\
& \times (e^{-\delta^2(t-v)} \mathbf{C}_2(v,t)(\mathbf{A}_1(0,1,0,0,v,t)\mathbf{1}) - \mathbf{U}_2(v,t)\mathbf{A}_2^2(0,1,0,0,v,t)\mathbf{1}), X_0\rangle) dv,
\end{aligned}$$

and

$$h_4(t) = e^{-\xi^1(0,t)L_0^1 - \xi^2(0,t)L_0^2} \langle (e^{-\delta^2 t} L_0^2 \mathbf{A}_1(1,1,0,0,0,t) - \mathbf{A}_2^2(1,1,0,0,0,t))\mathbf{1}, X_0\rangle,$$

with $\mathbf{C}_2(v,t) = \mathbf{diag}(\mathbf{a}^2)\mathbf{U}_2(v,t)$ *and* $\mathbf{U}_2(v,t) = \mathbf{diag}((e^{-a^{21}\xi^2(v,s)}, \cdots, e^{-a^{2N}\xi^2(v,s)})^*)$.

Proof The expected discounted spread payment is

$$\begin{aligned}
&\kappa E\left[\int_0^T e^{-\int_0^t r_s ds} 1_{\{\tau_2 > t\}} dt\right] = \kappa E\left[\int_0^T e^{-\int_0^t r_s ds} E\left[1_{\{\tau_2 > t\}} | \Im_t^X \vee \Im_t^L\right] dt\right] \\
&= \kappa E\left[\int_0^T \int_0^t L_v^1 e^{-\int_0^t r_s ds - \int_0^v L_u^1 du - \int_0^t L_u^2 du - a_v^2 \xi^2(v,t)} dv dt\right] \\
&+ \kappa E\left[\int_0^T e^{-\int_0^t (r_s + L_s^1 + L_s^2) ds} dt\right] \doteq \kappa(I_1 + I_2),
\end{aligned}$$

where the second equality follows from Proposition 3.3. Then by using Proposition 3.4, we have

$$I_2 = \int_0^T e^{-\xi^1(0,t)L_0^1 - \xi^2(0,t)L_0^2} \langle \mathbf{A}_1(1,1,0,0,0,t)\mathbf{1}, X_0\rangle dt. \tag{4.4}$$

In order to compute I_1, we use the "tower" property of conditional expectations and obatin

$$\begin{aligned}
I_1 &= \int_0^T \int_0^t E\left[L_v^1 e^{-\int_0^v (L_u^2+L_u^1+r_u)du - a_v^2\xi^2(v,t)} E\left[e^{-\int_v^t (r_u+L_u^2)du} | \Im_t^X \vee \Im_v^L\right]\right] dvdt \\
&= \int_0^T \int_0^t E\left[L_v^1 e^{-\int_0^v (L_u^2+L_u^1+r_u)du - \xi^2(v,t)L_v^2 - a_v^2\xi^2(v,t)} \langle \mathbf{A}_1(0,1,0,0,v,t)\mathbf{1}, X_v\rangle\right] dvdt \\
&= \int_0^T \int_0^t E\left[L_v^1 e^{-\int_0^v (L_u^2+L_u^1+r_u)du - \xi^2(v,t)L_v^2} \langle \mathbf{U}_2(v,t)(\mathbf{A}_1(0,1,0,0,v,t)\mathbf{1}), X_v\rangle\right] dvdt \\
&= \int_0^T \int_0^t e^{-\xi^1(0,v)L_0^1 - \xi^2(0,t)L_0^2} \Big\langle \Big(e^{-\delta^1 v} L_0^1 \mathbf{A}_1(1,1,0,\xi^2(v,t),0,v\Big) \\
&\quad - \mathbf{A}_2^1(1,1,0,\xi^2(v,t),0,v))\mathbf{U}_2(v,t)(\mathbf{A}_1(0,1,0,0,v,t)\mathbf{1}), X_0\Big\rangle dvdt, \qquad (4.5)
\end{aligned}$$

where the second equality follows from Proposition 3.4, and the last equality is a direct consequence of Corollary 3.1.

Now we turn to compute the expected discounted loss payment. From Proposition 3.3, we have

$$\begin{aligned}
&E\left[e^{-\int_0^{\tau_2} r_s ds} 1_{\{\tau_2 \le T\}}\right] = -E\left[\int_0^T e^{-\int_0^t r_s ds} dP\left(\tau_2 > t | \Im_t^X \vee \Im_t^L\right)\right] \\
&= -E\left[\int_0^T e^{-\int_0^t r_s ds} dP\left(\tau_1 > t, \tau_2 > t | \Im_t^X \vee \Im_t^L\right)\right] \\
&\quad -E\left[\int_0^T e^{-\int_0^t r_s ds} dP\left(\tau_2 > t, \tau_1 \le t | \Im_t^X \vee \Im_t^L\right)\right] \\
&= E\left[\int_0^T \int_0^t L_v^1 e^{-\int_0^t r_u du - \int_0^v L_u^1 du - \int_0^t L_u^2 du - a_v^2 \xi^2(v,t)} \left(L_t^2 + a_v^2 e^{-\delta^2(t-v)}\right) dvdt\right] \\
&\quad +E\left[\int_0^T L_t^2 e^{-\int_0^t (r_s + L_s^1 + L_s^2)ds} dt\right] \doteq T_1 + T_2.
\end{aligned}$$

An application of Corollary 3.1 yields

$$\begin{aligned}
T_2 &= \int_0^T e^{-\xi^1(0,t)L_0^1 - \xi^2(0,t)L_0^2} \Big\langle \Big(e^{-\delta^2 t} L_0^2 \mathbf{A}_1(1,1,0,0,0,t) \\
&\quad - \mathbf{A}_2^2(1,1,0,0,0,t)\Big)\mathbf{1}, X_0\Big\rangle dt. \qquad (4.6)
\end{aligned}$$

It remains to calculate T_1. Again using the "tower" property of conditional expectations yields

$$\begin{aligned}
T_1 &= E\Bigg[\int_0^T \int_0^t L_v^1 e^{-\int_0^v (r_u + L_u^2 + L_u^1)du - a_v^2 \xi^2(v,t)} \\
&\quad \times E\left[e^{-\int_v^t (r_u + L_u^2)du}\left(L_t^2 + a_v^2 e^{-\delta^2(t-v)}\right) | \Im_v^L \vee \Im_t^X\right] dvdt\Bigg] \\
&= \int_0^T \int_0^t E\Big[L_v^2 L_v^1 e^{-\int_0^v (r_u + L_u^2 + L_u^1)du - \xi^2(v,t)L_v^2} \langle e^{-\delta^2(t-v)} \mathbf{U}_2(v,t) \\
&\quad \times (\mathbf{A}_1(0,1,0,0,v,t)\mathbf{1}), X_v\rangle\Big] dvdt + \int_0^T \int_0^t E\Big[L_v^1 e^{-\int_0^v (r_u + L_u^2 + L_u^1)du - \xi^2(v,t)L_v^2} \\
&\quad \times \Big\langle e^{-\delta^2(t-v)} \mathbf{C}_2(v,t)(\mathbf{A}_1(0,1,0,0,v,t)\mathbf{1}) - \mathbf{U}_2(v,t)\mathbf{A}_2^2(0,1,0,0,v,t)\mathbf{1}, X_v\Big\rangle dvdt,
\end{aligned}$$

where the second equality is obtained from Proposition 3.4 and Corollary 3.1. Then, again using Corollary 3.1, we obtain

$$
\begin{aligned}
T_1 = \int_0^T \int_0^t & e^{-\xi^1(0,v)L_0^1-\xi^2(0,t)L_0^2}(\langle(e^{-(\delta^1+\delta^2)v}L_0^1L_0^2\mathbf{A}_1(1,1,0,\xi^2(v,t),0,v)\\
&-e^{-\delta^1 v}L_0^1\mathbf{A}_2^2(1,1,0,\xi^2(v,t),0,v)-e^{-\delta^2 v}L_0^2\mathbf{A}_2^1(1,1,0,\xi^2(v,t),0,v)\\
&+\mathbf{A}_3^{12}(1,1,0,\xi^2(v,t),0,v))e^{-\delta^2(t-v)}\mathbf{U}_2(v,t)(\mathbf{A}_1(0,1,0,0,v,t)\mathbf{1}),X_0\rangle\\
&+\langle(e^{-\delta^1 v}L_0^1\mathbf{A}_1(1,1,0,\xi^2(v,t),0,v)-\mathbf{A}_2^1(1,1,0,\xi^2(v,t),0,v))\\
&\times(e^{-\delta^2(t-v)}\mathbf{C}_2(v,t)(\mathbf{A}_1(0,1,0,0,v,t)\mathbf{1})-\mathbf{U}_2(v,t)\mathbf{A}_2^2(0,1,0,0,v,t)\mathbf{1}),X_0\rangle)dvdt.
\end{aligned}
\tag{4.7}
$$

Then equating the expected discounted loss payment with the expected discounted spread payment gives the result. □

Proposition 4.2 *The fair spread of the CDS default risk of the protection seller is given by*

$$\kappa_1 = \frac{(1-R)\int_0^T h_4(t)dt}{\int_0^T h_1(t)dt}, \tag{4.8}$$

where $h_1(t)$ and $h_4(t)$ are given in Proposition 4.1.

Proof The proof is similar to that of Proposition 4.1, so we just give an outline.

From Proposition 3.3, we can express the expected discounted spread payment as

$$\kappa_1 E\left[\int_0^T e^{-\int_0^t r_s ds}1_{\{\tau_1\wedge\tau_2>t\}}dt\right] = \kappa_1 E\left[\int_0^T e^{-\int_0^t(r_s+L_s^1+L_s^2)ds}dt\right].$$

The expected discounted loss payment is

$$
\begin{aligned}
E\left[e^{-\int_0^{\tau_2} r_s ds}1_{\{\tau_2\le T,\tau_2<\tau_1\}}\right] &= E\left[\int_0^T e^{-\int_0^t r_u du}1_{\{\tau_1\wedge\tau_2>t^-\}}dH_t^2\right]\\
= E\left[\int_0^T e^{-\int_0^t r_u du}1_{\{\tau_1\wedge\tau_2>t\}}L_t^2 dt\right] &= E\left[\int_0^T L_t^2 e^{-\int_0^t(r_s+L_s^1+L_s^2)ds}dt\right],
\end{aligned}
$$

where the second equality holds because $H_t^2-\int_0^{t\wedge\tau_2}\lambda_s^2 ds$ is a martingale.

Then using Eqs. 4.4 and 4.6 and equating the expected discounted loss payment with the expected discounted spread payment concludes the proof. □

4.2 First- and Second-to-Default CDSs on Two Underlyings

A kth-to-default swap, which is a commonly traded product of portfolio credit derivatives, is a bilateral contract between an insurance buyer and an insurance seller. The payment streams of this derivative depend on the default times of an underlying portfolio of n credit-risky assets. In this subsection, we evaluate the first- and second-to-default credit default swap (CDS) spreads on two underlyings.

Denote by $\tau_{(1)}=\tau_1\wedge\tau_2$ be the time of the first default. Denote by $\tau_{(2)}=\tau_1\vee\tau_2$ be the time of the second default. Assume the default dependence structure of the two underlyings is described by Eq. 2.3 in Section 2. Assume a unit notional and a constant recovery R. The buyer of a first-to-default (second-to-default) CDS pays a continuous spread $c_1(c_2)$ till the first (second) default occurs or till the maturity T of the contract if no credit event occurs before the maturity, in order to cover the loss of the protection buyer in the event of the credit event. Therefore, the fair spread of the kth-to-default swap c_k, $k = 1, 2$ should satisfy

$$c_k E\left[\int_0^T e^{-\int_0^t r_s ds} 1_{\{\tau_{(k)}>t\}} dt\right] = (1-R)E\left[e^{-\int_0^{\tau(k)} r_s ds} 1_{\{\tau_{(k)}\leq T\}}\right]. \tag{4.9}$$

The following results give the explicit expressions for c_1 and c_2.

Proposition 4.3 *The fair spread of the first-to-default swap on two underlyings is given by*

$$c_1 = \frac{(1-R)\int_0^T (h_4(t)+h_5(t))dt}{\int_0^T h_1(t)dt}, \tag{4.10}$$

where

$$h_5(t) = e^{-\xi^1(0,t)L_0^1 - \xi^2(0,t)L_0^2}\left\langle\left(e^{-\delta^1 t}L_0^1\mathbf{A}_1(1,1,0,0,0,t) - \mathbf{A}_2^1(1,1,0,0,0,t)\right)\mathbf{1}, X_0\right\rangle,$$

$h_1(t)$ and $h_4(t)$ are given in Propositions 4.1.

Proof The expected discounted spread payment is

$$c_1 E\left[\int_0^T e^{-\int_0^t r_s ds} 1_{\{\tau_{(1)}>t\}} dt\right] = c_1 E\left[\int_0^T e^{-\int_0^t (r_s+L_s^1+L_s^2)ds} dt\right] \doteq c_1 I_1,$$

where the expression for I_1 is given by Eq. 4.5.

The expected discounted loss payment is

$$E\left[e^{-\int_0^{\tau(1)} r_s ds} 1_{\{\tau_{(1)}\leq T\}}\right] = E\left[\int_0^T \left(L_t^1 + L_t^2\right) e^{-\int_0^t (r_s+L_s^1+L_s^2)ds} dt\right].$$

Then making use of Corollary 3.1 yields

$$E\left[e^{-\int_0^{\tau(1)} r_s ds} 1_{\{\tau_{(1)}\leq T\}}\right] = \int_0^T (h_4(t)+h_5(t))dt.$$

Therefore, Eq. 4.10 can be easily obtained from Eq. 4.9. The proof is finished. □

Proposition 4.4 *The fair spread of the second-to-default swap on two underlyings is given by*

$$c_2 = \frac{(1-R)\int_0^T (h_3(t)+h_7(t))dt}{\int_0^T (h_1(t)+h_2(t)+h_6(t))dt}, \tag{4.11}$$

where

$$\begin{aligned} h_6(t) = \int_0^t & e^{-\xi^1(0,t)L_0^1-\xi^2(0,v)L_0^2}\langle(e^{-\delta^2 v}L_0^2\mathbf{A}_1(1,1,\xi^1(v,t),0,0,v) \\ & -\mathbf{A}_2^2(1,1,\xi^1(v,t),0,0,v))\mathbf{U}_1(v,t)(\mathbf{A}_1(1,0,0,0,v,t)\mathbf{1}), X_0\rangle dv. \end{aligned}$$

$$\begin{aligned} h_7(t) = \int_0^t & e^{-\xi^1(0,t)L_0^1-\xi^2(0,v)L_0^2}(\langle(e^{-(\delta^1+\delta^2)v}L_0^1L_0^2\mathbf{A}_1(1,1,\xi^1(v,t),0,0,v) \\ & -e^{-\delta^1 v}L_0^1\mathbf{A}_2^2(1,1,\xi^1(v,t),0,0,v) - e^{-\delta^2 v}L_0^2\mathbf{A}_2^1(1,1,\xi^1(v,t),0,0,v) \\ & +\mathbf{A}_3^{12}(1,1,\xi^1(v,t),0,0,v))e^{-\delta^1(t-v)}\mathbf{U}_1(v,t)(\mathbf{A}_1(1,0,0,0,v,t)\mathbf{1}), X_0\rangle \\ & +\langle(e^{-\delta^2 v}L_0^2\mathbf{A}_1(1,1,\xi^1(v,t),0,0,v) - \mathbf{A}_2^2(1,1,\xi^1(v,t),0,0,v)) \\ & \times(e^{-\delta^1(t-v)}\mathbf{C}_1(v,t)(\mathbf{A}_1(1,0,0,0,v,t)\mathbf{1}) - \mathbf{U}_1(v,t)\mathbf{A}_2^1(0,1,0,0,v,t)\mathbf{1}), X_0\rangle)dv, \end{aligned}$$

with $\mathbf{C}_1(v,t) = \mathbf{diag}(\mathbf{a}^1)\mathbf{U}_1(v,t)$ *and* $\mathbf{U}_1(v,t) = \mathbf{diag}((e^{-a^{11}\xi^1(v,s)},\cdots,e^{-a^{1N}\xi^1(v,s)})^*)$, $h_1(t)$, $h_2(t)$ *and* $h_3(t)$ *are given in Propositions 4.1.*

Proof Dividing the event $\{\tau_2 > t\}$ into three mutually disjoint events: $\{\tau_1 > t, \tau_2 > t\}$, $\{\tau_1 > t, \tau_2 \le t\}$ and $\{\tau_1 \le t, \tau_2 > t\}$, then the expected discounted spread payment can be expressed as

$$
\begin{aligned}
& c_2 E\left[\int_0^T e^{-\int_0^t r_s ds} 1_{\{\tau_{(2)}>t\}} dt\right] = c_2 E\left[\int_0^T e^{-\int_0^t r_s ds} 1_{\{\tau_1>t,\tau_2>t\}} dt\right] \\
&+ c_2 E\left[\int_0^T e^{-\int_0^t r_s ds} 1_{\{\tau_1>t,\tau_2\le t\}} dt\right] + c_2 E\left[\int_0^T e^{-\int_0^t r_s ds} 1_{\{\tau_1\le t,\tau_2>t\}} dt\right] \\
&= c_2 E\left[\int_0^T \int_0^t L_v^1 e^{-\int_0^t r_s ds - \int_0^v L_u^1 du - \int_0^t L_u^2 du - a_v^2 \xi^2(v,t)} dv dt\right] \\
&+ c_2 E\left[\int_0^T \int_0^t L_v^2 e^{-\int_0^t r_s ds - \int_0^v L_u^2 du - \int_0^t L_u^1 du - a_v^1 \xi^1(v,t)} dv dt\right] \\
&+ c_2 E\left[\int_0^T e^{-\int_0^t (r_s + L_s^1 + L_s^2) ds} dt\right] \doteq c_2(I_1 + I_3 + I_2),
\end{aligned}
$$

where the formulas for I_1 and I_2 are given by Eqs. 4.5 and (4.4), respectively.

Similar to deriving Eq. 4.5, we can obtain

$$
\begin{aligned}
I_3 = &\int_0^T \int_0^t e^{-\xi^1(0,t)L_0^1 - \xi^2(0,v)L_0^2} \langle (e^{-\delta^2 v} L_0^2 \mathbf{A}_1(1,1,\xi^1(v,t),0,0,v) \\
&-\mathbf{A}_2^2(1,1,\xi^1(v,t),0,0,v))\mathbf{U}_1(v,t)(\mathbf{A}_1(1,0,0,0,v,t)\mathbf{1}), X_0\rangle dv dt.
\end{aligned}
$$

Now we turn to compute the expected discounted loss payment. Using the "tower property" of conditional expectations and Proposition 3.3, we have

$$
\begin{aligned}
& E\left[e^{-\int_0^{\tau(2)} r_s ds} 1_{\{\tau(2)\le T\}}\right] = -E\left[\int_0^T e^{-\int_0^t r_s ds} dP\left(\tau_{(2)} > t | \Im_t^X \vee \Im_t^L\right)\right] \\
&= -E\left[\int_0^T e^{-\int_0^t r_s ds} dP\left(\tau_1 > t, \tau_2 > t | \Im_t^X \vee \Im_t^L\right)\right] \\
&- E\left[\int_0^T e^{-\int_0^t r_s ds} dP\left(\tau_1 > t, \tau_2 \le t | \Im_t^X \vee \Im_t^L\right)\right] \\
&- E\left[\int_0^T e^{-\int_0^t r_s ds} dP\left(\tau_2 > t, \tau_1 \le t | \Im_t^X \vee \Im_t^L\right)\right] \\
&= E\left[\int_0^T \int_0^t L_v^1 e^{-\int_0^t r_s ds - \int_0^v L_u^1 du - \int_0^t L_u^2 du - a_v^2 \xi^2(v,t)} \left(L_t^2 + a_v^2 e^{-\delta^2(t-v)}\right) dv dt\right] \\
&+ E\left[\int_0^T \int_0^t L_v^2 e^{-\int_0^t r_s ds - \int_0^v L_u^2 du - \int_0^t L_u^1 du - a_v^1 \xi^1(v,t)} \left(L_t^1 + a_v^1 e^{-\delta^1(t-v)}\right) dv dt\right] \doteq T_1 + T_2,
\end{aligned}
$$

where the expression for T_1 is given by (4.7).

Similar to deriving (4.7), we have

$$
\begin{aligned}
T_2 = & \int_0^T \int_0^t e^{-\xi^1(0,t)L_0^1-\xi^2(0,v)L_0^2}(\langle(e^{-(\delta^1+\delta^2)v}L_0^1L_0^2\mathbf{A}_1(1,1,\xi^1(v,t),0,0,v)\\
&-e^{-\delta^1 v}L_0^1\mathbf{A}_2^2(1,1,\xi^1(v,t),0,0,v)-e^{-\delta^2 v}L_0^2\mathbf{A}_2^1(1,1,\xi^1(v,t),0,0,v)\\
&+\mathbf{A}_3^{12}(1,1,\xi^1(v,t),0,0,v))e^{-\delta^1(t-v)}\mathbf{U}_1(v,t)(\mathbf{A}_1(1,0,0,0,v,t)\mathbf{1}),X_0\rangle\\
&+\langle(e^{-\delta^2 v}L_0^2\mathbf{A}_1(1,1,\xi^1(v,t),0,0,v)-\mathbf{A}_2^2(1,1,\xi^1(v,t),0,0,v))\\
&\times(e^{-\delta^1(t-v)}\mathbf{C_1}(v,t)(\mathbf{A}_1(1,0,0,0,v,t)\mathbf{1})-\mathbf{U}_1(v,t)\mathbf{A}_2^1(0,1,0,0,v,t)\mathbf{1}),X_0\rangle)dvdt.
\end{aligned}
$$

Therefore, equating the expected discounted spread payment with the expected discounted loss payment ends the proof. □

5 Numerical Results

This section numerically investigates how the parameters influence the CDS spreads. For ease of illustration, we consider $N = 2$, that is X switches between only two states, where state e_1 and state e_2 represent a "good" macro-economic condition and a "bad" macro-economic condition, respectively. Let $T = 5$, $R = 0.4$, $\mathbf{r} = (0.05, 0.02)^*$, $\delta^i = 10$, $\mathbf{L}_0^1 = \mathbf{L}_0^2 = (0.01, 0.05)^*$, $\mathbf{a}^1 = \mathbf{a}^2 = (0.005, 0.015)^*$, $\boldsymbol{\rho}_1 = \boldsymbol{\rho}_2 = (2, 6)^*$, $\boldsymbol{\rho}_3 = (1, 3)^*$. The densities $\mathbf{f}^1$ and $\mathbf{f}^2$ are given by $f^{i1}(x) = \beta_1 e^{-\beta_1 x}, x > 0$, and $f^{i2}(x) = \beta_2 e^{-\beta_2 x}, x > 0$, with $(\beta_1, \beta_2)^* = (200, 50)^*$. Assume $q_{11} = q_{22} = -q$.

Figures 1 and 2 present the impact of the model parameters on the single-name CDS spread without counterparty risk. In these figures, we see that the spread in the case with the "good" economy at time $t = 0$ is much lower. We also see that a larger q results in a larger spread if $X_0 = e_1$. This is because the probability of switching to the bad economy increases as q increases. On the other hand, if we start at the "bad" economy, the spreads decrease as q increases. This is mainly due to the increasing probability of switching to the good economy. In Fig. 1, we see that the spread increases with the arrival rate of the shock

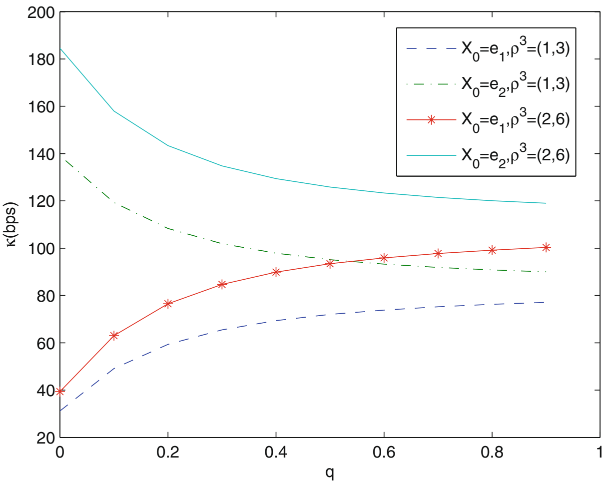

Fig. 1 Relationship between κ and q for different $\boldsymbol{\rho}^3$

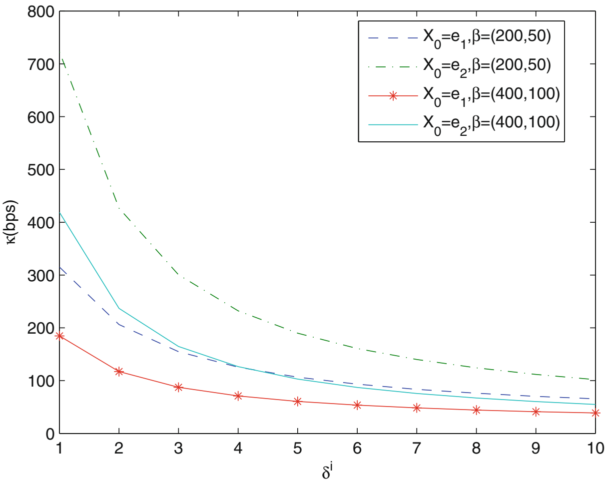

Fig. 2 Relationship between κ and δ^i for different $\boldsymbol{\beta}$, $q = 0.3$

events with other parameters being fixed. Since an increase in the arrival rate leads to a higher frequency that the intensities jump upward, the default probability for the reference entity increases. In Fig. 2, we observe that the impact of δ^i on the spread is very obvious with a larger δ^i corresponding to a lower spread. This may be explained by the fact that the time period that the default intensity goes back to the previous level of intensity immediately after major events occur will be shorten as ai increases. We can also see the spread increases with the jump amount of the shock events with other parameters being fixed, since an increase in the jump amount leads to the increasing of the default intensity.

Figures 3 and 4 present the impact of the parameters on the single-name CDS spread with counterparty risk. The curves in Figs. 3 and 4 are similar to those in Figs. 1 and 2.

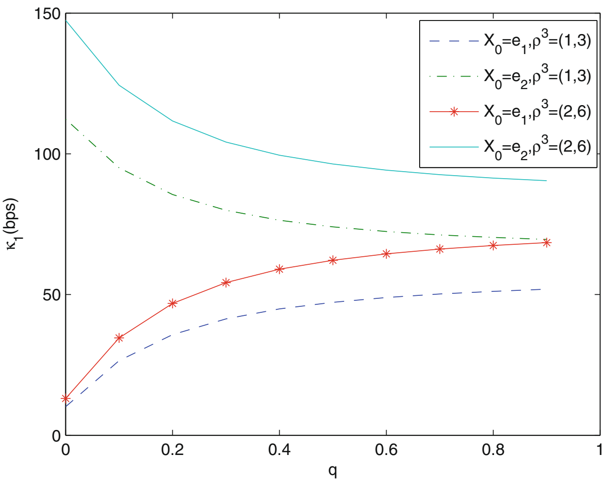

Fig. 3 Relationship between κ_1 and q for different $\boldsymbol{\rho^3}$

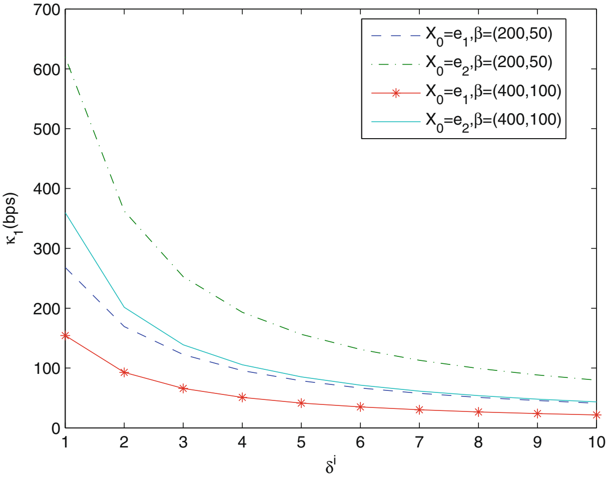

Fig. 4 Relationship between κ and δ^i for different $\boldsymbol{\beta}$, $q = 0.3$

Figures 1 and 4 indicate that the spread with counterparty risk is lower than the one without counterparty risk. This is consistent with the financial intuition.

Figures 5 and 6 present the impact of parameters on the CDS spread difference $\kappa - \kappa_1$. Figure 5 shows that the difference increases with q when $X_0 = e_1$, while it decreases with q when $X_0 = e_2$. We also see that a larger $\kappa - \kappa_1$ corresponds to a larger arrival rate. In Fig. 6, we observe that the impact of the parameter δ^i on $\kappa - \kappa_1$ is very obvious, and that the difference decreases with δ^i. We can also see a larger $\kappa - \kappa_1$ corresponds to a larger jump amount.

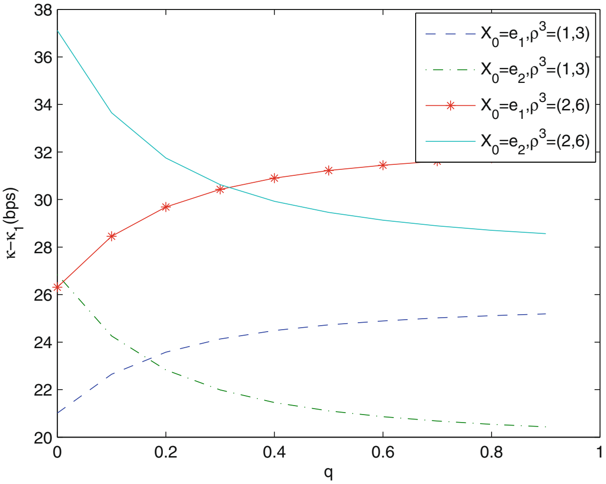

Fig. 5 Relationship between $\kappa - \kappa_1$ and q for different $\boldsymbol{\rho}^{\mathbf{3}}$

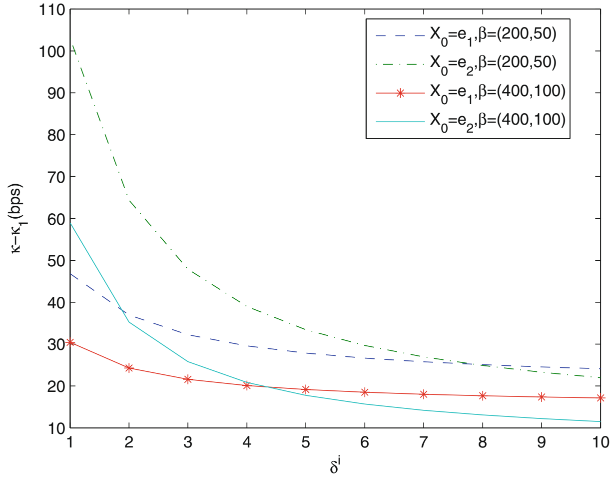

Fig. 6 Relationship between $\kappa - \kappa_1$ and δ^i for different $\boldsymbol{\beta}$, $q = 0.3$

Figures 7, 8, 9 and 10 present the impact of the model parameters on the spreads of the first and second-to-default basket swaps on two underlyings. The curves in Figs. 7–10 are similar to those in Figs. 1–4. Comparing Figs. 7 and 8 with Figs. 1 and 2, we see that the spread of the first-to-default basket swap is much higher than the single-name CDS spread. This is in line with the stylized feature: the first-to-default swap spread written on a portfolio of n reference names increases with n. Comparing Figs. 7 and 8 with Figs. 9 and 10, we can observe the first-to-default CDS spread is much larger than the second-to-default spread, also in line with stylized features.

We remark that since we focus on providing a theoretical pricing model, we just make some numerical analysis without doing the calibration in this paper. One thing on our future research agenda is to use the credit market CDS spreads to empirically test our model.

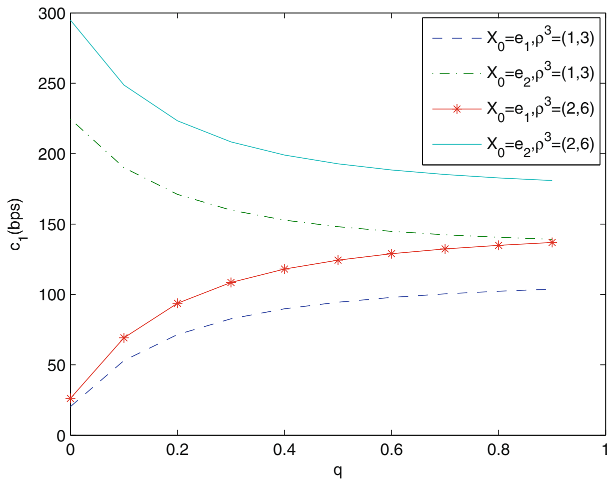

Fig. 7 Relationship between c_1 and q for different $\boldsymbol{\rho}^{\mathbf{3}}$

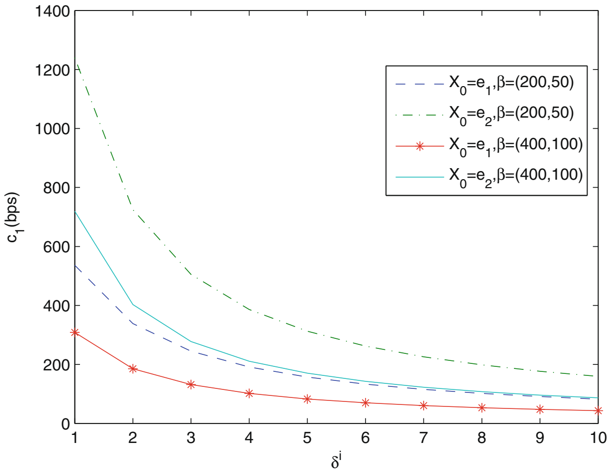

Fig. 8 Relationship between c_1 and δ^i for different $\boldsymbol{\beta}$, $q = 0.3$

Giesecke et al. (2011b) suggest there exist three regimes and obtain the transitional probability by making analysis on the corporate bond market over the course of the last 150 years. Therefore, the generator of the Markov chain can be borrowed from Giesecke et al. (2011b). The parameters θ may be obtained according to

$$\theta = \mathbf{arg}\min_{\hat{\theta}} \sum_{T \in \{T_1, \cdots, T_k\}} \frac{(c(T, \hat{\theta}) - c(T))^2}{c(T)^2},$$

where $T_1, \cdots, T_k$ are different maturities and $c(T)$ is the CDS spread observed from the market. We will use some good methods of parameter estimation to obtain the parameter estimates in the future's research.

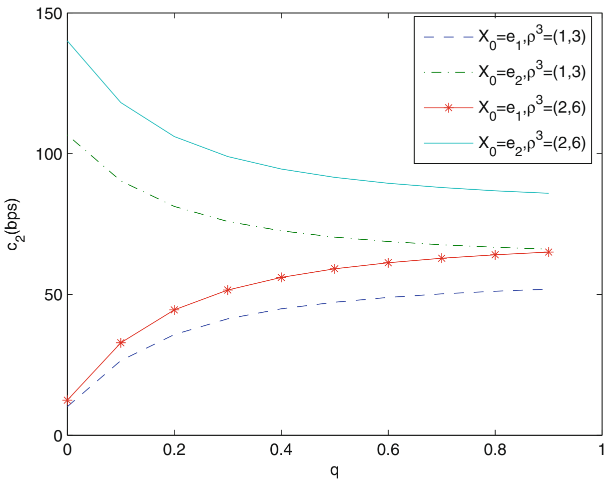

Fig. 9 Relationship between c_2 and q for different $\boldsymbol{\rho}^{\mathbf{3}}$

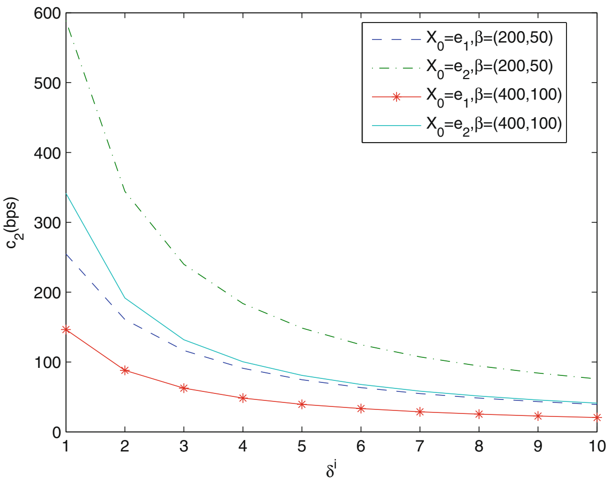

Fig. 10 Relationship between c_2 and δ^i for different $\boldsymbol{\beta}$, $q = 0.3$

6 A Multi-dimensional Contagion Model

In this section, we extend the two-dimensional interacting intensities model Eq. 2.3 to a multi-dimensional one.

Consider a group of m firms in the market. Denote τ_i to be the default time of the ith firm. Assume that the default intensity of τ_i is given by

$$\lambda_t^i = L_t^i + \sum_{j=1, j\neq i}^{m} a_{\tau_j}^i e^{-\delta^i (t-\tau_j)} 1_{\{\tau_j \leq t\}}, i = 1, 2, \cdots, m, \tag{6.1}$$

where $a_{\tau_j}^i = \langle \mathbf{a}^i, X_{\tau_j} \rangle$ for a constant vector $\mathbf{a}^i = (a^{i1}, \cdots, a^{iN})^*$ with $a^{il} > 0$ for each $i = 1, 2, \cdots, m, l = 1, \cdots, N$, and L_t^i is a regime-switching shot noise process given by

$$L_t^i = L_0^i e^{-\delta^i t} + \int_0^t e^{-\delta^i (t-s)} dJ_s^i, i = 1, 2, \cdots, m.$$

Here $\delta^1, \delta^2, \cdots, \delta^m$ are positive constants; $L_0^i = \langle \mathbf{L}_0^i, X_0 \rangle$, where $\mathbf{L}_0^i = \left(L_0^{i1}, \cdots, L_0^{iN}\right)^*$ with $L_0^{il} > 0$ for each $i = 1, 2, \cdots, m, l = 1, 2, \cdots, N$; and $J_t^i = \sum_{j=1}^{N_i(t)+N_{m+1}(t)} Y_j^i$, where $N_1(t), N_2(t), \cdots, N_m(t)$ and $N_{m+1}(t)$ are mutually conditionally independent regime-switching Poisson processes with intensities given by $\rho_i(s) = \langle \boldsymbol{\rho}_i, X_s \rangle$ for constant vectors $\boldsymbol{\rho}_i = \left(\rho_i^1, \cdots, \rho_i^N\right)^*, i = 1, 2, \cdots, m+1$ with $\rho_i^j > 0$, for each $i = 1, 2, \cdots, m+1, l = 1, \cdots, N$; Assume that given the path of the Markov chain X, the sequences $\left\{Y_1^1, Y_2^1, \cdots\right\}, \left\{Y_1^2, Y_2^2, \cdots\right\}, \cdots, \left\{Y_1^m, Y_2^m, \cdots\right\}$ are mutually independent and independent of $N_1(t), N_2(t), \cdots, N_{m+1}(t)$. Furthermore, given the path of the Markov chain X, we assume that for each $i = 1, 2, \cdots, m$ the jump sizes Y_j^i, $j = 1, 2, \cdots$ are mutually independent and identically distributed with a common conditional density f_t^i concentrated on $(0, \infty)$, where $f_t^i(.) = \langle \mathbf{f}^i(.), X_t \rangle$, with $\mathbf{f}^i(.) = (f^{i1}(.), \cdots, f^{iN}(.))^*$. Then the process $\left(L_t^1, L_t^2, \cdots, L_t^m\right)$ is an m-dimensional regime-switching shot noise process with common jumps.

Denote the filtration by

$$\Im_t = \Im_t^X \vee \Im_t^L \vee \Im_t^1 \vee \Im_t^2 \vee \cdots \vee \Im_t^m,$$

where $\Im_t^L = \Im_t^{L^1} \vee \Im_t^{L^2} \vee \cdots \vee \Im_t^{L^m}$, and $\Im_t^i = \sigma\left(H_u^i : 0 \le u \le t\right)$, with $H_u^i = 1_{\{\tau_i \le u\}}$, $\Im_t^{L^i} = \sigma\left(L_u^i : 0 \le u \le t\right)$, $i = 1, 2, \cdots, m$.

Similar to the proof of Proposition 3.4, we can obtain the following result.

Proposition 6.1 *For $c^i \ge 0$ and $d^i > 0$, $i = 1, 2, \cdots, m$, we have*

$$E\left[e^{-\int_t^T \sum_{i=1}^m c^i L_s^i ds - \sum_{i=1}^m d^i L_T^i} X_T | \Im_t^X \vee \Im_t^L\right]$$
$$= e^{-\sum_{i=1}^m (c^i \xi^i(t,T) + d^i e^{-\delta^i (T-t)}) L_t^i} \left\langle \mathbf{B}_1(c^1, \cdots, c^m, d^1, \cdots, d^m, t, T), X_t \right\rangle, \tag{6.2}$$

where

$$\xi^i(t, T) = (1 - e^{-\delta^i (T-t)})/\delta^i, i = 1, 2, \cdots, m$$

and the matrix $\mathbf{B}_1(c^1, \cdots, c^m, d^1, \cdots, d^m, t, T)$ solves

$$\frac{\partial \mathbf{B}_1}{\partial t} + (Q + \mathbf{diag}(\tilde{\mathbf{F}}_t(c^1, \cdots, c^m, d^1, \cdots, d^m))\mathbf{B}_1(c^1, \cdots, c^m, d^1, \cdots, d^m, t, T) = 0,$$

with boundary condition

$$\mathbf{B}_1(c^1, \cdots, c^m, d^1, \cdots, d^m, T, T) = \mathbf{I}.$$

Here, $\mathbf{I}$ is an $N \times N$ identity matrix and $\tilde{\mathbf{F}}_s$ is an N-dimensional vector with the jth component given by

$$\tilde{F}_s^j(c^1, \cdots, c^m, d^1, \cdots, d^m) = \sum_{i=1}^m \rho_i^j \left(g_s^{ij}(c^i, d^i) - 1\right) + \rho_{m+1}^j \left(\prod_{i=1}^m g_s^{ij}(c^i, d^i) - 1\right),$$

and

$$g_s^{ij}(c^i, d^i) = \int_0^\infty e^{-(c^i \xi^i(s,T) + d^i e^{-\delta^i (T-s)})x} f^{ij}(x) dx, \qquad i = 1, 2, \cdots, m, \; j = 1, \cdots, N.$$

Furthermore,

$$E\left[e^{-\int_t^T \left(\sum_{i=1}^m c^i L_s^i + r_s\right) ds - \sum_{i=1}^m d^i L_T^i} | \Im_t^X \vee \Im_t^L\right]$$
$$= e^{-\sum_{i=1}^m (c^i \xi^i(t,T) + d^i e^{-\delta^i (T-t)}) L_t^i} \langle \mathbf{B}_1(c^1, \cdots, c^m, d^1, \cdots, d^m, t, T)\mathbf{1}, X_t \rangle. \tag{6.3}$$

Proof Since the proof is similar to the one of Proposition 3.4, we omit it. □

In order to derive the joint survival probability $P(\tau_1 > t_1, \tau_2 > t_2, \cdots, \tau_m > t_m)$, we follow the idea of change of measure adopted in Collin-Dufresne et al. (2004) and Giesecke and Zhu (2013). For each $i = 1, \cdots, m$, define the probability measures P^i that is absolutely continuous with respect to P by Eq. 3.1, and define the probability measures $P^{1,\cdots,m}$ by

$$\frac{dP^{1,\cdots,m}}{dP}|_{\Im_t} = \prod_{i=1}^{m} 1_{\{\tau_i > t\}} \exp\left(\int_0^t \lambda_s^i ds\right) \doteq \prod_{i=1}^{m} \eta_t^i. \tag{6.4}$$

Similar to the proofs of Propositions 3.1–3.2, we can show under $P^{1,\cdots,m}$, the Markov chain X_t and the jump process L_t^i still have the same distributions as those under P.

Therefore, for any $t \geq 0$, changing the measure from P to $P^{1,\cdots,m}$ yields

$$\begin{aligned} P(\tau_1 > t, \tau_2 > t, \cdots, \tau_m > t) &= E^{1,\cdots,m}\left[e^{-\int_0^t (L_s^1 + L_s^2 + \cdots + L_s^m) ds}\right] \\ &= e^{-\sum_{i=1}^{m} \xi^i(0,t) L_0^i} \langle \mathbf{B}_1(1, \cdots, 1, 0, \cdots, 0, 0, t)\mathbf{1}, X_0\rangle, \end{aligned}$$

where the last equality is obtained by using Proposition 6.1 and the fact that the distributions of the Markov chain X_t and the jump process L_t^i under $P^{1,\cdots,m}$ are the same as those under P.

However, it is difficult to give the explicit expression for $P(\tau_1 > t_1, \tau_2 > t_2, \cdots, \tau_m > t_m)$. One possible method may be to consider a relation between the $(m-1)$-dimensional conditional joint density and the m-dimensional conditional joint density. Since we can obtain the two-dimensional conditional joint density for τ_1 and τ_2 from Proposition 3.3, the three-dimensional conditional joint density for τ_1, τ_2 and τ_3 can be derived by using the idea of "change of measure."

Here we only consider the case $t_1 \leq t_2 \leq t_3$. For $t_1 \leq t_2 \leq t_3$ and any event $A \in \Im_{t_3}^X \vee \Im_{t_3}^L$, using the "tower property" of conditional expectations yields

$$E\left[1_{\{A\}} E\left[1_{\{\tau_1 > t_1, \tau_2 > t_2, \tau_3 > t_3\}} | \Im_{t_3}^X \vee \Im_{t_3}^L\right]\right] = E[1_{\{\tau_1 > t_1, \tau_2 > t_2, \tau_3 > t_3\}} 1_{\{A\}}].$$

Then changing the measure from P to P^3 yields

$$\begin{aligned} &E[1_{\{\tau_1 > t_1, \tau_2 > t_2, \tau_3 > t_3\}} 1_{\{A\}}] \\ &= E^3\left[1_{\{\tau_1 > t_1, \tau_2 > t_2\}} e^{-\int_0^{t_3} (L_u^3 + 1_{\{\tau_1 \leq u\}} a_{\tau_1}^3 e^{-\delta^3(u-\tau_1)} + 1_{\{\tau_2 \leq u\}} a_{\tau_2}^3 e^{-\delta^3(u-\tau_2)}) du} 1_{\{A\}}\right]. \end{aligned}$$

Since under P^3, the default intensities of τ_1 and τ_2 are given by Eq. 2.3, we can obtain $E\left[1_{\{\tau_1 > t_1, \tau_2 > t_2, \tau_3 > t_3\}} | \Im_{t_3}^X \vee \Im_{t_3}^L\right]$ by using the two-dimensional conditional joint density of τ_1 and τ_2. Therefore, the three-dimensional conditional joint density $f\left(t_1, t_2, t_3 | \Im_{t_3}^X \vee \Im_{t_3}^L\right)$ for $t_1 \leq t_2 \leq t_3$ can be obtained by differentiating $E[1_{\{\tau_1 > t_1, \tau_2 > t_2, \tau_3 > t_3\}} | \Im_{t_3}^X \vee \Im_{t_3}^L]$ with respect to t_1, t_2 and t_3. In the future's research, we shall investigate the relation between the $(m-1)$-dimensional conditional joint density and the m-dimensional conditional joint density.

7 Conclusions

In this paper, extending Errais et al. (2010), we consider a two-dimensional regime-switching affine jump intensity model to analyze a single-name CDS spread with and without counterparty default risk, the first and second-to-default CDS spreads on two underlyings. Our model includes both self-excited and externally excited jumps. The default dependence structure we construct stems from three sources. First, the intensities of the two firms are both affected by a Markov chain describing macro-economy. Second, default dependence arises from common jumps in the intensities modeled by a regime-switching compound Poisson process. Finally, inter-dependent default structure arises from default contagion.

In order to obtain the CDS spreads, we follow the idea of "change of measure" to solve the looping structure of default intensities. Therefore, under the survival measure, the default intensity is modeled by a regime-switching shot-noise process, which is a special case of a regime-switching affine jump diffusion process and can be well used to measure the impact on the default intensity of exogenous shock events. We show the distributions of the regime-switching shot noise processes under the new measures are the same as those under the original measure. Furthermore, by using a martingale method, we obtain the joint Laplace transform of the regime-switching shot noise processes. Based on these results, the single-name CDS spreads with and without counterparty default risk, the first- and second-to-default CDS spreads on two underlyings can be represented in terms of fundamental matrix solutions of linear, matrix-valued, ordinary differential equations. Two things on our future research agenda are to empirically test our model using statistical data from CDS markets and to apply our model to calculate unilateral credit adjustment valuation for a CDS contract.

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Appendix

Proof of Proposition 3.1

Proof Since M_t is an R^N-valued martingale under P, it suffices to prove that $M_t\eta_t^i$ is an $\Im-$martingale under P. From Itô's formula, we have

$$M_t\eta_t^i = M_0 + \int_0^t \eta_{s-}^i dM_s + \int_0^t M_{s-}d\eta_s^i + \sum_{s\leq t}\Delta M_s\Delta\eta_s^i.$$

Since M_t and η_t^i are both $\Im-$martingales under P, it remains to show the last term vanishes. Note that, given $\Delta\eta_s^i \neq 0$ at $s = \tau_i$, the summation in the last term above will be zero provided that $\Delta M_{\tau_i} = 0$, a.s.. In fact, $P(\Delta M_t \neq 0) = P(\Delta X_t \neq 0) = 0$ for any fixed time $t > 0$. And from the definition of τ_i given by $\tau_i = \min\{t > 0 : \int_0^t \lambda_s^i ds \geq E_i\}$, we have $P(\Delta X_{\tau_i} \neq 0) = E[E[1_{\{\Delta X_{\tau_i}\neq 0\}}|X_s]] = E\left[E\left[\sum_{j=1}^{\infty} 1_{\{\tau_i=T_j\}}|X_s\right]\right]$, where $0 < T_1 < T_2 < \cdots$ denote the transition times of X. Since $E\left[\sum_{j=1}^{\infty} 1_{\{\tau_i=T_j\}}|X_s\right] = E\left[\sum_{j=1}^{\infty} 1_{\{\int_0^{T_j}\lambda_s^i ds=E_i\}}|X_s\right] = 0$, then $\Delta M_{\tau_i} = \Delta X_{\tau_i} = 0$, a.s. The proof is completed. $\square$

Proof of Proposition 3.2

Proof The proof of Proposition 3.2 is similar to the one of Proposition 3.1.

We shall prove that $\overline{N}_i(t)\eta_t^i$ and $\overline{M}_i(t)\eta_t^i$ are both $\Im-$martingales under P. From Itô's formula, we have

$$\overline{N}_i(t)\eta_t^i = \overline{N}_i(0) + \int_0^t \eta_{s-}^i d\overline{N}_i(s) + \int_0^t \overline{N}_i(s-)d\eta_s^i + \sum_{s\le t} \Delta\overline{N}_i(s)\Delta\eta_s^i,$$

and

$$\overline{M}_i(t)\eta_t^i = \overline{M}_i(0) + \int_0^t \eta_{s-}^i d\overline{M}_i(s) + \int_0^t \overline{M}_i(s-)d\eta_s^i + \sum_{s\le t} \Delta\overline{M}_i(s)\Delta\eta_s^i.$$

Since $\overline{N}_i(t)$, $\overline{M}_i(t)$ and η_t^i are all $\Im-$martingales under P, it remains to show $\Delta\overline{N}_i(\tau_i) = 0$, $\Delta\overline{M}_i(\tau_i) = 0$, a.s.. In fact,

$$P(\Delta\overline{N}_i(\tau_i)\neq 0) = P(\Delta N_i(\tau_i)\neq 0) = E[E[1_{\{\Delta N_i(\tau_i)\neq 0\}}|X_s]] = E\left[E\left[\sum_{j=1}^{\infty} 1_{\{\tau_i = T_j\}}|X_s\right]\right],$$

and

$$P(\Delta\overline{M}_i(\tau_i)\neq 0) = P(\Delta J_{\tau_i}^i \neq 0) = E\left[E\left[1_{\{\Delta J_{\tau_i}^i\neq 0\}}|X_s\right]\right] = E\left[E\left[\sum_{j=1}^{\infty} 1_{\{\tau_i = T_j\}}|X_s\right]\right],$$

where $0 < T_1 < T_2 < \cdots$ denote the jump times of N_i. Since $E\left[\sum_{j=1}^{\infty} 1_{\{\tau_i = T_j\}}|X_s\right] = E\left[\sum_{j=1}^{\infty} 1_{\left\{\int_0^{T_j}\lambda_s^i ds = E_i\right\}}|X_s\right] = 0$, then $\Delta\overline{N}_i(\tau_i) = 0$, $\Delta\overline{M}_i(\tau_i) = 0$, a.s.. The proof is completed. □

Proof of Proposition 3.3

Proof For any event $A \in \Im_s^X \vee \Im_s^L$, using the "tower property" of conditional expectations yields

$$E\left[1_{\{A\}}E\left[1_{\{\tau_1>s,\tau_2>s\}}|\Im_s^X\vee\Im_s^L\right]\right] = E[1_{\{\tau_1>s,\tau_2>s\}}1_{\{A\}}].$$

Then changing the measure from P to P^1 yields

$$\begin{aligned}
E[1_{\{\tau_1>s,\tau_2>s\}}1_{\{A\}}] &= E\left[1_{\{\tau_2>s\}}\eta_s^1 e^{-\int_0^s \lambda_u^1 du}1_{\{A\}}\right] \\
&= E^1\left[1_{\{\tau_2>s\}}e^{-\int_0^s L_u^1 du}1_{\{A\}}\right] \\
&= E^1\left[E^1\left[1_{\{\tau_2>s\}}|\Im_s^X\vee\Im_s^L\right]e^{-\int_0^s L_u^1 du}1_{\{A\}}\right] \\
&= E^1\left[e^{-\int_0^s (L_u^1+L_u^2)du}1_{\{A\}}\right] = E\left[e^{-\int_0^s (L_u^1+L_u^2)du}1_{\{A\}}\right],
\end{aligned}$$

where the last second equality holds because τ_2 has the intensity L_t^2 under measure P^1, and the last equality holds because the distributions of L_t^i and X_t under measure P^1 are the same as those under P. The proof of Eq. 3.2 is finished.

The proof of Eq. 3.3 is similar. For any event $A \in \Im_s^X \vee \Im_s^L$,

$$
\begin{aligned}
& E\left[E\left[1_{\{\tau_1>s,t<\tau_2\le s\}}|\Im_s^X \vee \Im_s^L\right]1_{\{A\}}\right] \\
= \; & E[1_{\{\tau_1>s,t<\tau_2\le s\}}1_{\{A\}}] = E\left[\eta_s^1 1_{\{t<\tau_2\le s\}}e^{-\int_0^s (L_u^1+1_{\{\tau_2\le u\}}a_{\tau_2}^1 e^{-\delta^1(u-\tau_2)})du}1_{\{A\}}\right] \\
= \; & E^1\left[1_{\{t<\tau_2\le s\}}e^{-\int_0^s L_u^1 du - a_{\tau_2}^1\xi^1(\tau_2,s)}1_{\{A\}}\right] \\
= \; & E^1\left[E^1\left[1_{\{t<\tau_2\le s\}}e^{-\int_0^s L_u^1 du - a_{\tau_2}^1\xi^1(\tau_2,s)}|\Im_T^X \vee \Im_T^L\right]1_{\{A\}}\right] \\
= \; & E^1\left[\int_t^s L_v^2 e^{-\int_0^v L_u^2 du - a_v^1\xi^1(v,s) - \int_0^s L_u^1 du}dv 1_{\{A\}}\right] \\
= \; & E\left[\int_t^s L_v^2 e^{-\int_0^v L_u^2 du - a_v^1\xi^1(v,s) - \int_0^s L_u^1 du}dv 1_{\{A\}}\right],
\end{aligned}
$$

where the last second equality holds because τ_2 has the intensity L_t^2 under measure P^1, and the last equality holds because the distributions of L_t^i and X_t under measure P^1 are the same as those under P. The proof of Eq. 3.3 is finished.

The proof of Eq. 3.4 is similar to the one of Eq. 3.3, so we omit it. □

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